Comparison of Numerical Error Estimators for Eddy Current Problems solved by FEM

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In the domain of field computation with the finite element method, choosing the mesh refinement is an important step to obtain an accurate solution. In order to evaluate the quality of the mesh, a posteriori error estimators are frequently used. In this communication we propose to analyze and to compare residual and equilibrated error estimators for eddy current problems.

***Index Terms*— Electromagnetic fields, Error estimator, Finite element analysis,**

I. Introduction

The Finite Element Method is widely used to solve eddy current problems. Today one of the challenges is to evaluate the quality of the solution with the help of error estimators. Some of them [1-3] based on *a posteriori* error analysis give an estimate of the spatial error distribution that can be used in the remeshing step. In this family the residual-type error estimators can be used, their drawback is that the gap between the error and the estimator is unknown. A way to handle with this is to use equilibrated *a posteriori* error estimators built from the property of non-verification of equilibrium equations. They give the spatial error distribution too and, in addition, they provide very sharp and global error bounds.

In this communication we present residual and equilibrated error estimators for eddy current problems solved with **A**-φ and **T**-Ω harmonic formulations. A residual estimator is proposed for each of them. In the case of the equilibrated estimator two approaches are proposed. These estimators are compared and discussed.

II. Numerical Development

A. Weak Formulation

Let us consider a domain $\mathcal{D}$ of boundary Γ_B. $\mathcal{D}$ is divided into three subdomains: $\mathcal{D}_s$ with the source term corresponding to the current density $\mathbf{J}_s$, a conducting part $\mathcal{D}_c$ and the non-conducting one $\mathcal{D}_{nc}$. The boundary condition on Γ_B is $\mathbf{B}.\mathbf{n}=0$ with **B** the magnetic flux density. To study our problem both well known **A**-φ and **T**-Ω formulations will be considered.

In this paper the harmonic **A**-φ formulation is developed but the transposition to magnetic **T**-Ω formulation is easy [4]. From Maxwell's equations with the boundary condition on Γ_B, the weak formulation can be written as:

$$\begin{aligned}&(\mu^{-1}\mathbf{curl}\mathbf{A},\mathbf{curl}\mathbf{A}')_{\mathcal{D}}+(\sigma(j\omega\mathbf{A}+\mathbf{grad}\varphi,\mathbf{A}')_{\mathcal{D}_c}=(\mathbf{J}_s,\mathbf{A}')_{\mathcal{D}}\\&(\sigma(j\omega\mathbf{A}+\mathbf{grad}\varphi),\mathbf{grad}\varphi')_{\mathcal{D}_c}=0\end{aligned}\qquad(1)$$

where μ represents the magnetic permeability, σ the conductivity, and ω the pulsation. **A'** and φ' are the test functions. Now, the domain $\mathcal{D}$ is discretized with a mesh τ_h made of tetrahedra denoted $\mathcal{T}$ and the facet $\mathcal{F}$. The discretisation of the weak formulation (1) takes the form:

$$(\mu^{-1}\mathbf{curl}\mathbf{A}_h,\mathbf{curl}\mathbf{A}_h')_{\mathcal{D}}+(\sigma(j\omega\mathbf{A}_h+\mathbf{grad}\varphi_h),\mathbf{A}_h')_{\mathcal{D}_c}=(\mathbf{J}_s,\mathbf{A}_h')_{\mathcal{D}}$$

$$(\sigma(j\omega\mathbf{A}_h+\mathbf{grad}\varphi_h),\mathbf{grad}\varphi_h')_{\mathcal{D}_c}=0\qquad(2)$$

Here, the subscript h means that the corresponding functions belong to finite dimensional spaces associated to the mesh τ_h.

B. Definitions of reliability and efficiency

From a mathematical point of view, the reliability and the efficiency properties [3] prove that the estimator is equivalent to the error and justify its use in an adaptive mesh refinement framework. The reliability is defined by:

$$\varepsilon\le C_1\eta\qquad(3)$$

and the efficiency by:

$$C_2\eta_{\mathcal{T}}\le\varepsilon_{\mathcal{P}(\mathcal{T})}\qquad(4)$$

where C_1 and C_2 are two positive constants which only depend on the data of the problem but not on the mesh size h. η and $\eta_{\mathcal{T}}$ are respectively the global error estimator and the local error estimator in the element $\mathcal{T}$. In the same way ε and $\varepsilon_{\mathcal{P}(\mathcal{T})}$ represent the global error in the whole domain and the local error in the patch of the element $\mathcal{P}(\mathcal{T})$ [3].

C. Residual error estimator

The local error estimator on a tetrahedron $\mathcal{T}$ is defined by [2]:

$$\eta_{\mathcal{T}}^2=\eta_{\mathcal{T};1}^2+\eta_{\mathcal{T};2}^2+\eta_{\mathcal{T};3}^2+\sum_{\mathcal{F}\subset\partial\mathcal{T}}(\eta_{\mathcal{F};1}^2+\eta_{\mathcal{F};2}^2),\qquad(5)$$

where :

$$\eta_{\mathcal{T};1}=h_{\mathcal{T}}\left\|\mathbf{J}_{sh}-\mathbf{curl}(\mu^{-1}\mathbf{curl}\mathbf{A}_h)-\sigma(j\omega\mathbf{A}_h+\mathbf{grad}\varphi_h)\right\|_{\mathcal{T}}\qquad(6)$$

$$\eta_{\mathcal{T};2}=h_{\mathcal{T}}\left\|\mathrm{div}(\sigma(j\omega\mathbf{A}_h+\mathbf{grad}\varphi_h))\right\|_{\mathcal{T}}\qquad(7)$$

$$\eta_{\mathcal{T};3}=h_{\mathcal{T}}\left\|\mathbf{J}_s-\pi_h\mathbf{J}_s\right\|_{\mathcal{T}}\qquad(8)$$

$$\eta_{\mathcal{F};1}=h_{\mathcal{F}}^{1/2}\left\|\left[\mathbf{n}\wedge\mu^{-1}\mathbf{curl}\mathbf{A}_h\right]_{\mathcal{F}}\right\|_{\mathcal{F}}\qquad(9)$$

$$\eta_{\mathcal{F};2}=h_{\mathcal{F}}^{1/2}\left\|\left[\sigma(j\omega\mathbf{A}_h+\mathbf{grad}\varphi_h).\mathbf{n}\right]_{\mathcal{F}}\right\|_{\mathcal{F}}\qquad(10)$$

we denote $h_{\mathcal{T}}$ and $h_{\mathcal{F}}$ the diameter of the element and the facet respectively. (6) and (7) evaluate, for each element, the error on the volumic residual. The expression (8) gives the discretisation error of the source term. Finally, (9) and (10) evaluate the jumps of the normal component of the induced current density $\mathbf{J}_h$ and the jump of the tangential component of magnetic field $\mathbf{H}_h$ through a facet between two elements. From the local estimation $\eta_{\mathcal{T}}$ the global error estimation η_{Res} is obtained by a discrete sum on the mesh.

D. Equilibrated error estimator

For example from the **A**-φ formulation [5] with finite element method we obtain a pair of admissible fields (magnetic flux density $\mathbf{B}_h$ and electric field $\mathbf{E}_h$). In the same way the **T**-Ω formulation gives two admissible fields ($\mathbf{J}_h$ and $\mathbf{H}_h$). From these fields, and the non-verification of equilibrium equations at the discrete level, it is possible to define the local error estimator on a tetrahedron $\mathcal{T}$ so that [5]:

$$\eta^2_{\mathcal{T}-\text{Eq.}} = \left\| \mu^{1/2}\left(\mathbf{H}_h - \mu^{-1}\mathbf{B}_h\right)\right\|^2_{\mathcal{T}} + \left\|(\sigma\omega)^{-1/2}\left(\mathbf{J}_h - \sigma\mathbf{E}_h\right)\right\|^2_{\mathcal{T}} \qquad (11)$$

It can be shown that there is a link between the local estimator and the local error [5]:

$$\eta^2_{\mathcal{T}-\text{Eq.}} \le 2(\varepsilon^2_{\mathcal{T}\text{-}\mathbf{A}-\varphi} + \varepsilon^2_{\mathcal{T}-\mathbf{T}-\Omega}) \qquad (12)$$

where $\varepsilon_{\mathcal{T}\text{-}\mathbf{A}\text{-}\varphi}$ and $\varepsilon_{\mathcal{T}\text{-}\mathbf{T}\text{-}\Omega}$ are the errors due to respectively **A**-φ and **T**-Ω formulations. As previously from the local estimation $\eta_{\mathcal{T}\text{-Eq.}}$ the global error estimation $\eta_{\text{Eq.}}$ is obtained by a discrete sum on the mesh. As shown in [5] there is direct link between the estimator and the error so that:

$$\eta^2_{\text{Eq.}} = \sum_{\mathcal{T}\in\tau_h} \eta^2_{\mathcal{T}-\text{Eq.}} = (\varepsilon^2_{\mathbf{A}-\varphi} + \varepsilon^2_{\mathbf{T}-\Omega}) \qquad (13)$$

In order to determine the admissible fields as explained before we can use both formulations (**A**-φ and **T**-Ω) in this case we will denote the estimator $\eta_{\text{Eq.dual}}$. Another solution consists in using the **A**-φ formulation to obtain $\mathbf{B}_h$ and $\mathbf{E}_h$ and then to construct locally the two other admissible fields $\mathbf{J}_h$ and $\mathbf{H}_h$ [6]. This estimator will be denoted $\eta_{\text{Eq.Const}}$.

III. NUMERICAL APPLICATIONS

In order to evaluate the estimators' efficiency a structure composed of a coil between two conductive plates has been studied (figure 1). The coil is fed by a sinusoidal voltage at the frequency of 50Hz. To compare the error estimators four regular meshes have been used between 4,500 and 2,290,000 elements.

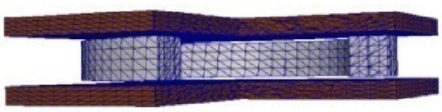

Figure 1: Mesh of the studied structure

For different meshes figure 2 shows the evolution of the magnetic energy obtained by both classical formulations. In the same way figure 3 shows the power losses in the plates. From these figures we observe the convergence of the estimators with respect to the number of elements. To compare the proposed estimators figure 4 shows the evolution of the error estimators in function of the meshes. We observe that all the estimators have a slope a little bit less of -1/3 which is the reference in finite elements for regular solutions. This figure shows also that the results of both equilibrated estimators $\eta_{\text{Eq.dual}}$ and $\eta_{\text{Eq.Const}}$ are similar. Conversely due to the unknown constant in (3) and (4) for the residual estimators, there results are distant from the equilibrated estimator. Nevertheless the estimator $\eta_{\text{Res-}\mathbf{T}\text{-}\Omega}$ gives some results near both equilibrated estimators.

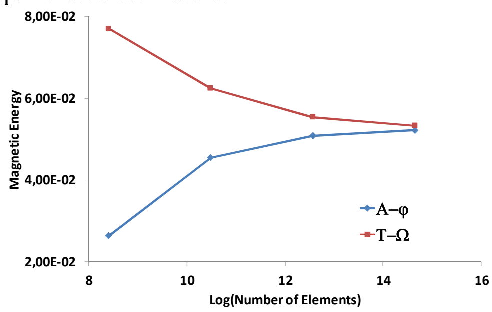

Figure 2: Evolution of the magnetic energy for different meshes

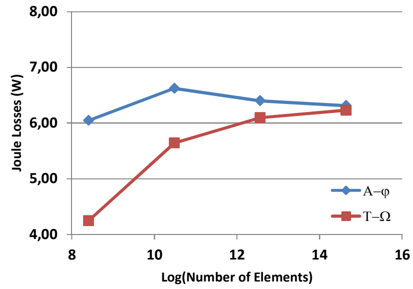

Figure 3: Evolution of Joule losses different meshes

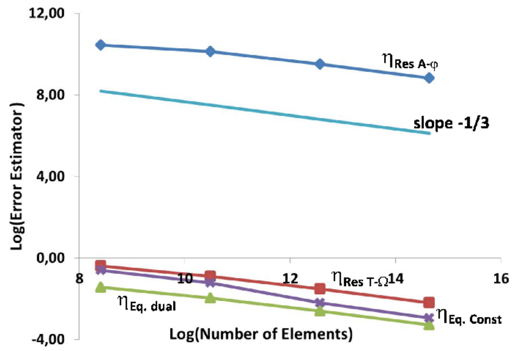

Figure 4: Comparison of residual and equilibrated estimators

In conclusion two families of error estimators are proposed for classical formulations of harmonic eddy current problems. For a given structure some comparisons are done as a function of the meshes. All proposed estimators are in good agreement with the expected convergence behavior.

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